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LETTER TO THE EDITOR

Renormalization-group results for the three-state Potts model

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Abstract. A lower-bound renormalization transformation of the type introduced by Kadanoff is applied to the three-state Potts model. For both $d = 2$ and $d = 3$ the Kadanoff transformation predicts a continuous transition. Values for the critical exponents and the critical temperature are reported. The consistency of the $d = 2$ results with series estimates gives us some confidence in the predictions for $d = 3$.

The three-state Potts (1952) model has the Hamiltonian

$$\mathscr{H} = -J\sum_{\langle ij\rangle}\delta_{\sigma_i\sigma_j} - \sum_{i\alpha}\zeta_\alpha\delta_{\sigma_i\alpha} \tag{1}$$

where i denotes a lattice site and where the spins σ and the index α take three values. Straley and Fisher (1973) have pointed out that if the model exhibits a continuous transition in zero field ($\zeta = 0$), the critical point is actually an anomalous tricritical point in the T–ζ phase diagram. A Landau expansion of the free energy and mean-field theory (Mittag and Stephen 1974) predict a first-order rather than a continuous transition in zero field. Baxter (1973) has shown that the q-state Potts model on a $d = 2$ square lattice has a first-order transition for $q > 4$ and a higher-order transition for $q \leqslant 4$. Series analyses for $q = 3$ have been carried out by Ditzian (1974), Ditzian and Oitmaa (1974), Enting (1974), Kim and Joseph (1975), Straley (1974), Straley and Fisher (1973), and others. There is general agreement that the series indicate a continuous transition for $d = 2$, but for $d = 3$ opinion is divided. A renormalization-group calculation by Golner (1973) using Wilson's approximate recursion relation predicts a first-order transition for $d = 3$. For a summary of results using the ϵ expansion we refer to Priest and Lubensky (1976).

Position-space renormalization-group methods have been applied to the $d = 2$ square-lattice three-state Potts model by Berker and Wortis (1976) and by Dasgupta (1976 and unpublished results mentioned in Berker and Wortis 1976). We learned of Dasgupta's work, which is similar to our own, during the preparation of this manuscript, which includes results for the $d = 3$ BCC lattice as well. Ashkin–Teller–Potts models with $q \neq 3$ have been considered by Dasgupta (1976), Harris *et al* (1975), and Knops (1975). The $d = 2$, $q = 3$ calculations predict a continuous transition and yield critical exponents consistent with the series analyses. The results reported in this letter were obtained using a lower-bound position-space transformation of the type invented by

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Kadanoff (Kadanoff 1975, Kadanoff *et al* 1976). Kadanoff's transformation has the advantages that it can readily be extended to $d > 2$ and that it is extremely successful in predicting the critical exponents of the Ising model for $d = 2$, 3, 4. The reason why Kadanoff's method works so well for the Ising model is not fully understood. In particular it is not clear what determines the choice between the several critical fixed points appearing in Kadanoff's method because of a special symmetry (Burkhardt 1976a, Knops 1976). Secondly it appears that certain correction terms arising from a more systematic variational procedure should not be taken into account (Kadanoff *et al* 1976, Knops 1976). In the present letter we pragmatically apply Kadanoff's method to the Potts model following the procedure which yields the best results for the Ising critical exponents.

The three-state Potts model is a special case of the Blume–Emery–Griffiths (Blume *et al* 1971) model. In obtaining the results reported here, we used the spin-1 lower-bound transformation applied to the Blume–Emery–Griffiths model by Burkhardt (1976b), restricting the three variational parameters p to the one-parameter subspace $\boldsymbol{p} = p_0\ (1, -2, 3)$, so that the transformation preserves the symmetry of the zero-field Potts model under permutations of the three states.

As in Kadanoff (1975), Kadanoff *et al* (1976) and Burkhardt (1976a,b), we perform an exact decimation transformation on the initial Hamiltonian to enter an invariant subspace of the Kadanoff transformation in which one only considers interactions symmetric in all the spin variables. In the invariant subspace the lower-bound transformation has high- and low-temperature fixed points and a Potts tricritical fixed point. The values $p_0^* = 0{\cdot}8446$ ($d = 2$) and $0{\cdot}4528$ ($d = 3$) maximize the free energy at the Potts fixed point and were used in calculating critical temperatures and exponents. For the $d = 2$ square lattice we find $J/k_BT_0 = 1{\cdot}037$ for the transition temperature, which compares favourably with the exact value $1{\cdot}005$ obtained by Potts from a duality transformation. For the $d = 3$ BCC lattice our result is $J/k_BT_0 = 0{\cdot}4012$.

In the position-space renormalization-group approach a first-order transition is associated with a discontinuity fixed point† (Nienhuis and Nauenberg 1975) with eigenvalue $y = d$ for each eigen-operator conjugate to a discontinuous order parameter. For both $d = 2$ and $d = 3$ the eigenvalues we find all satisfy $y < d$ and correspond to a second- rather than a first-order transition. Table 1 shows the eigenvalues we calculate and the related critical exponents. The eigenvalue $y_{4\mathrm{p}}$ corresponds to a temperature-like direction in the Potts space. The other relevant eigenvalues, which are all doubly degenerate as a consequence of the Potts symmetry (Berker and Wortis 1976), correspond to fields breaking the Potts symmetry. The eigenvalues $y_{1\mathrm{p}}$ and $y_{2\mathrm{p}}$ are connected with the crossover to critical behaviour whereas $y_{3\mathrm{p}}$ and $y_{6\mathrm{p}}$ are associated with the crossover to normal tricritical behaviour. (The distinction between normal and special tricritical behaviour was first made by Straley and Fisher (1973).) Some $d = 2$ series results are listed for comparison. Except in the case of α_{p} the series results are in excellent agreement with our results. We suspect that α_{p} is closer to our calculated value than the series value of Straley and Fisher (1973) since the latter is inconsistent with the scaling laws $2-\alpha = \beta(\delta+1) = \gamma(\delta+1)/(\delta-1)$. We refer to Berker and Wortis (1976) for comparison with the exponents obtained in other $d = 2$ position-space renormalization-group calculations.

That the lower-bound transformation is quite successful in predicting the $d = 2, 3$ Ising exponents and the $d = 2$ Potts exponents gives us some confidence in the results

† This is in contrast to the situation encountered in the ϵ expansion where no true fixed point is expected in the symmetric space when the transition is first order (Amit 1976).

Table 1. Relevant eigenvalues and critical exponents of the Potts tricritical fixed point. We follow the notation of Berker and Wortis (1976). The numbers in parentheses denote series results.

Eigenvalue, exponent	$d=2$	$d=3$
$y_{2p}=y_{1p}$	1·872	2·511
y_{4p}	1·202	2·071
$y_{6p}=y_{3p}$	0·4610	0·8867
α_p	0·3365 ($0{\cdot}05\pm0{\cdot}10^{a}$, $0{\cdot}286\pm0{\cdot}02^{b}$)	0·5514
δ_p	14·68 ($15{\cdot}0\pm0{\cdot}04^{c}$)	5·131
β_p	0·1061 ($0{\cdot}10\pm0{\cdot}01^{a}$, $0{\cdot}105\pm0{\cdot}005^{c}$)	0·2363
γ_p	1·451 ($1{\cdot}5\pm0{\cdot}2^{a}$, $1{\cdot}45\pm0{\cdot}15^{c}$, $1{\cdot}42\pm0{\cdot}05^{d}$)	0·9761
$\gamma_p+\Delta_p$	3·009 ($3{\cdot}00\pm0{\cdot}1^{d}$)	2·188

a Straley and Fisher (1973)
b Zwanzig and Ramshaw (1973, preprint mentioned in Kim and Joseph 1975).
c Enting (1974).
d Kim and Joseph (1975).

for the $d=3$ Potts model. Note that for $d=3$ the Potts tricritical exponents α_p, δ_p, β_p and γ_p are close to the mean-field values $\frac{1}{2}$, 5, $\frac{1}{4}$, and 1 of the normal tricritical exponents predicted for $d\geqslant3$ by Riedel and Wegner (1972) using renormalization-group arguments. Many of our $d=3$ exponents are consistent with the series estimates of Straley (1974); see also Enting (1974).

In the Potts subspace of variational parameters $\boldsymbol{p}=p_0(1,-2,3)$ we have also found fixed points associated with normal critical, normal tricritical, and first-order transitions in the Blume–Emery–Griffiths model. These results will be reported in a longer article.

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